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On Weak Chebyshev Subspaces. II. Continuous Selection for the Metric Projection and Extension to Mairhuber's Theorem

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The main result in this paper is the characterization of all n-dimensional weak Chebyshev Z subspaces of $C(Q)$ for which the metric projection has a continuous selection. It is also shown that if $n \geqslant 3$ and P_N has a continuous selection, then Q should be homeomorphic to a subset of R. © 1991 Academic Press, Inc.

1. INTRODUCTION

The closed subset A of the normed linear space X is said to be proximinal in X, if for each $x \in X$ there is $y \in A$ such that $d(x, A) = \|x - y\|$, where $d(x, A)$ is the distance from x to A; that is,

$$d(x, A) = \inf\{\|x - y\|; y \in A\}.$$

For the proximinal set A in X, the set-valued function $P_A : X \to 2^A$ defined by $P_A(x) = \{y \in A; \|x - y\| = d(x, A)\}$ is called the metric projection from X onto A, and if there is a continuous function $f: X \to A$ such that $f(x) \in P_A(x)$ for each $x \in X$, then f is called a continuous selection for the metric projection P_A.

"Q is a totally ordered space" means that Q is a totally ordered set and the topology defined on it is the order topology. If Q is a locally compact totally ordered space, then $C_0(Q)$ is the Banach space of all continuous real-valued functions defined on Q and "vanishing at infinity"; that is, if $f \in C_0(Q)$, then for all $\varepsilon > 0$, the set $\{q \in Q; |f(q)| \geqslant \varepsilon\}$ is compact. If Q is compact then $C_0(Q)$ is denoted $C(Q)$. The norm defined on $C_0(Q)$ and $C(Q)$ is the uniform norm; that is, $\|f\| = \sup\{|f(q)|; q \in Q\}$. The subspace

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N of $C_0(Q)$ is called a Z-subspace if no $g \neq 0$ in N vanishes on a nonempty open subset of Q. Any subset of the real numbers is totally ordered, and any proper subset of the circle is totally ordered. Another very important totally ordered space is the "interval with split points" (for definition see Brown [1, 2]; also see Example 4.11 in this paper). Totally ordered spaces have a very strong relation with the existence of a continuous selection for the metric projection. Brown [1] proved that if Q is any compact Hausdorff space and $C(Q)$ contains a finite dimensional Z-subspace N of dimension at least two such that the metric projection P_N has a continuous selection, then either Q is homeomorphic to a subset of the circle or Q is homeomorphic to a subset of an interval with split points.

If Q is a locally compact totally ordered space, then the n-dimensional subspace N of $C_0(Q)$ is called a Chebyshev subspace if each $g \neq 0$ in N has no more than $(n-1)$ zeros. N is called a weak Chebyshev subspace if for each basis $\{g_1, g_2, ..., g_n\}$ of N, $x_1 < x_2 < \cdots < x_n$ in Q, and $y_1 < y_2 < \cdots < y_n$ in Q,

$$\det[g_i(x_j)] \cdot \det[g_i(y_j)] \geqslant 0.$$

Jones and Karlovitz [4], Deutsch, Nurnberger, and Singer [3], and Kamal [5] studied other equivalent properties of the weak Chebyshev subspaces. One of these properties is the following:

For each $f \in C_0(Q)$ there is $g \in N$ such that $\|f-g\| = d(f, N)$ and $(f-g)$ equioscillates at $(n+1)$ points of Q; that is, there are $x_1 < x_2 < \cdots < x_{n+1}$ in Q and $\varepsilon = \pm 1$, such that

$$(-1)^i (f-g)(x_i) = \varepsilon \|f-g\|, \qquad \text{for} \quad i = 1, 2, ..., n+1.$$

This property is related to the existence of a continuous selection for the metric projection $P_N: C_0(Q) \to 2^N$. This relation can be seen in the following theorem:

1.1. THEOREM. *Let Q be a locally compact totally ordered space, let N be an n-dimensional subspace of $C_0(Q)$, and let P_N be the metric projection from $C_0(Q)$ onto N. If for each $f \in C_0(Q)$ there is a unique $g_f \in P_N(f)$ such that $(f-g_f)$ equioscillates at $(n+1)$ points, then the mapping $\psi: C_0(Q) \to N$ defined by $\psi(f) = g_f$ is a continuous selection for the metric projection P_N.*

The proof of this theorem is easy and can be obtained from the proof of the special case when Q is a compact real interval; that was done by Nurnberger and Sommer [8].

1.2. DEFINITION. Let Q be a locally compact totally ordered space, let N be an n-dimensional subspace of $C_0(Q)$, and let P_N be the metric projection from $C_0(Q)$ onto N. The subspace N may or may not possess one of the following properties:

awc_1: Each $g \neq 0$ in N has at most n distinct zeros

awc_2: For each $f \in C_0(Q)$ there is a unique $g \in P_N(f)$, such that $(f-g)$ equioscillates at $(n+1)$ points.

By Theorem 1.1, if N has the property awc_2 then the metric projection P_N has a continuous selection. In the case when N is a weak Chebyshev subspace, each $f \in C_0(Q)$ has at least one $g \in P_N(f)$ such that $(f-g)$ equioscillates at $(n+1)$ points of Q, so in order to show that the metric projection P_N from $C_0(Q)$ onto the n-dimensional weak Chebyshev subspace N of $C_0(Q)$ has a continuous selection, it is enough to show that, for each $f \in C_0(Q)$, there is at most one $g \in P_N(f)$ such that $(f-g)$ equioscillates at $(n+1)$ points. Using the properties of the real intervals, Nurnberger and Sommer [8] proved that the properties awc_1 and awc_2 are equivalent for any n-dimensional weak Chebyshev subspace N of $C[a, b]$, where $[a, b]$ is a compact real interval. Nurnberger [6] obtained the same result for any n-dimensional weak Chebyshev subspace of $C_0(Q)$, where Q is any locally compact subset of the real numbers. However, his proof is very difficult and depends very strongly on the properties of the real numbers, so it cannot be generalized any more.

In this paper the author studies the property awc_1 and its relation with the existence of a continuous selection for the metric projection in the general case when Q is any locally compact (resp. compact) totally ordered space. In Section 2, the author studies the properties of the order topology on Q that are related to the existence of the property awc_1 in some n-dimensional weak Chebyshev subspaces of $C_0(Q)$. These properties are not algebraic, and they are satisfied by some spaces that are not homeomorphic to subsets of the real numbers. In Section 3, the author uses some of these properties to prove that the properties awc_1 and awc_2 are equivalent on any n-dimensional weak Chebyshev subspace of $C_0(Q)$, where Q is any locally compact totally ordered space. The proof is very simple and natural. Combining this result with some other results, it is shown that if N is a finite-dimensional weak Chebyshev Z-subspace of $C(Q)$, then the metric projection P_N has a continuous selection if and only if N has the property awc_1. This result gives a full characterization for those finite-dimensional weak Chebyshev Z-subspaces of $C(Q)$ for which the metric projection P_N has a continuous selection.

The natural question that one may ask is whether the property awc_1 is satisfied by some n-dimensional weak Chebyshev subspaces of $C(Q)$, when Q is not homeomorphic to any subset of the real numbers. The answer is an extension to Mairhuber's theorem. Mairhuber's theorem asserts that if there is a Chebyshev subspace of $C(Q)$ of finite dimension not less than two, then Q is homeomorphic to a subset of the circle. The proof of Mairhuber's theorem can be found in Singer [10]. In Section 4, it is shown that if Q is a compact totally ordered space, $C(Q)$ contains an n-dimensional weak Chebyshev subspace that has the property awc_1, and $n \geqslant 3$, then Q is homeomorphic to a subset of R. In the case when $\dim N = 2$, an example will be given to show that this result fails. However, if $\dim N = 2$ and there is $x_0 \in Q$ such that $g(x_0) = 0$ for each $g \in N$, then the result holds. Combining this result with other results from Section 3, it is shown also that if Q is a compact totally ordered space and $C(Q)$ contains a finite-dimensional weak Chebyshev Z-subspace of dimension not less than three, and the metric projection P_N has a continuous selection, then Q is homeomorphic to a subset of R. In the case when the dimension of this subspace is 2, an example will given to show that this result fails.

The rest of this section will cover some definitions and known results that will be used later in this paper. In this paper "Q is a totally ordered space" means that Q is a totally ordered set with the order topology defined on it. The intervals $[x, y]$, (x, y) in Q and the terminologies $-\infty$ and $+\infty$ have their ordinary meaning. If Q is a locally compact totally ordered space, then $f \in C_0(Q)$ is said to "oscillate weakly" (resp. "oscillate") at k points of Q if there are $x_1 < x_2 < \cdots < x_k$ in Q and $\varepsilon = \pm 1$ such that $(-1)^i \varepsilon f(x_i) \geqslant 0$ (resp. $(-1)^i \varepsilon f(x_i) > 0$) for all $i = 1, 2, ..., k$. f is said to "equioscillate" at k points of Q if there are $x_1 < x_2 < \cdots < x_k$ in Q, and $\varepsilon = \pm 1$ such that $(-1)^i f(x_i) = \varepsilon \|f\|$ for all $i = 1, 2, ..., k$. If N is an n-dimensional subspace of Q, then the points $x_1, x_2, ..., x_k$ are said to be "N-independent" if the linear functionals $\hat{x}_1, \hat{x}_2, ..., \hat{x}_k$ defined by $\hat{x}_i(g) = g(x_i)$ are linearly independent in N^*, the dual space of N.

The proof of the following lemma is elementary:

1.3. Lemma. *Let Q be a locally compact Hausdorff space, and let N be an n-dimensional subspace of $C_0(Q)$. The points $x_1, x_2, ..., x_k$, $k \leqslant n$ in Q are N-independent if and only if for each $\alpha_1, \alpha_2, ..., \alpha_k$ in R (the set of real numbers), there is $g \in N$ such that $g(x_i) = \alpha_i$ for each $i = 1, 2, ..., k$.*

1.4. Definition. Let Q and N be as in Lemma 1.3. The distinct points $x_1, x_2, ..., x_k$ in Q are called "N-totally dependent" if there are $\lambda_1, \lambda_2, ..., \lambda_k$ in R with $\lambda_i \neq 0$ for each i, such that $\sum_{i=1}^{k} \lambda_i \hat{x}_i = 0$, where $\hat{x}_i$ is the linear functional in N^* defined by x_i.

An N-totally dependent subset $\{x_1, x_2, ..., x_k\}$ of Q need not be a "minimal dependent" subset of Q with respect to N, but in Section 2, it will be shown that if N has the property awc_1, and $1 \leqslant k \leqslant n$, then any N-totally dependent subset $\{x_1, x_2, ..., x_k\}$ of Q is a minimal dependent subset with respect to N. Obviously each N-dependent subset of Q contains a nonempty N-totally dependent subset.

1.5. THEOREM (Kamal [5]). *Let Q be a locally compact totally ordered space that contains at least $(n+1)$ points, and let N be an n-dimensional subspace of $C_0(Q)$. Then the following properties are equivalent*:

wc_1: *Each $g \neq 0$ in N has at most $(n-1)$ changes of sign; that is, no g in N oscillates at $(n+1)$ points or more in Q.*

wc_2: *N is a weak Chebyshev subspace of $C_0(Q)$.*

wc_3: *For each $x_1 < x_2 < \cdots < x_{n-1}$ in Q, there is $g \neq 0$ in N such that $g(x_i) = 0$ for $i = 1, 2, ..., n-1$, and*

$$(-1)^i g(x) \geqslant 0 \text{ for } x \in (x_i, x_{i+1}), \qquad \text{for} \quad i = 1, 2, ..., n-1,$$

where $x_0 = -\infty$ and $x_n = +\infty$.

wc_4: *For each $f \in C_0(Q)$ there is $g \in N$ such that $\|f - g\| = d(f, N)$, and $(f - g)$ equioscillates at $(n+1)$ points in Q.*

2. THE PROPERTY awc_1

In this section some simple results will be obtained to clarify the relation between the property awc_1 and the order topology on Q. These results will be used in Section 3 and Section 4 to obtain the main results.

2.1. LEMMA. *Let Q be a locally compact Hausdorff space, let N be an n-dimensional subspace of $C_0(Q)$ that has the property* awc_1, *and let $\{x_1, x_2, ..., x_k\}$, $1 \leqslant k \leqslant n$, be an N-totally dependent subset of Q. Then for each $y_{k+1}, ..., y_{k+\imath}$ in $Q \backslash \{x_1, ..., x_k\}$, where $k \leqslant k + \imath \leqslant n+1$, and each $i_0 \in \{1, 2, ..., k\}$ the points*

$$x_1, ..., x_{i_0-1}, x_{i_0+1}, ..., x_k, y_{k+1}, ..., y_{k+\imath}$$

are N-independent.

Proof. If Q consists of exactly n elements then the proof is obvious. So without loss of generality one might assume that Q contains at least $n+1$ elements, and $k + \imath = n+1$.

The set $\{x_1, x_2, ..., x_k\}$ is N-totally dependent, so there are $\lambda_1, ..., \lambda_{i_0-1}, \lambda_{i_0+1}, ..., \lambda_k$ in R with $\lambda_i \neq 0$ for each i, such that

$$g(x_{i_0}) = \sum_{\substack{i=1 \\ i \neq i_0}}^{k} \lambda_i g(x_i) \qquad \text{for each} \quad g \in N.$$

Let $\{x_1, ..., x_{i_0-1}, x_{i_0+1}, ..., x_k, y_{k+1}, ..., y_{n+1}\} = \{z_1, z_2, ..., z_n\}$, and assume that the points $z_1, z_2, ..., z_n$ are N-dependent. Then there are $j_0 \in \{1, 2, ..., n\}$ and $\mu_1, \mu_2, ..., \mu_{j_0-1}, \mu_{j_0+1}, ..., \mu_n$ in R such that

$$g(z_{j_0}) = \sum_{\substack{i=1 \\ i \neq j_0}}^{n} \mu_i g(z_i) \qquad \text{for each} \quad g \in N.$$

Since $\dim N = n$, it follows that there is $g \neq 0$ in N, such that $g(s) = 0$ for each $s \in \{z_1, ..., z_{j_0-1}, z_{j_0+1}, ..., z_n\}$. But then $g(z_{j_0}) = 0$ and therefore $g(x_{i_0}) = 0$, so g has more than n zeros in Q, which contradicts the fact that N has the property awc_1.

2.2. COROLLARY. *Let Q, N, and $\{x_1, ..., x_k\}$ be as in Lemma 2.1. Then any proper nonempty subset of $\{x_1, ..., x_k\}$ is N-independent.*

In Theorem 2.3, the notation "$x = \sum_{i=1}^{k} \lambda_i x_i$," means that

$$g(x) = \sum_{i=1}^{k} \lambda_i g(x_i) \qquad \textit{for each} \quad g \in N.$$

If $\{g_1, ..., g_n\}$ is a basis for N, and $x_1, x_2, ..., x_n$ are in Q, then $\det[g_i(x_j)]$ *will be denoted by $|x_1, x_2, ..., x_n|$; that is,*

$$|x_1, x_2, ..., x_n| = \begin{vmatrix} g_1(x_1) & g_1(x_2) & \cdots & g_1(x_n) \\ g_2(x_1) & g_2(x_2) & \cdots & g_2(x_n) \\ \vdots & \vdots & & \vdots \\ g_n(x_1) & g_n(x_2) & \cdots & g_n(x_n) \end{vmatrix}.$$

2.3. THEOREM. *Let Q be a locally compact totally ordered space, let N be an n-dimensional weak Chebyshev subspace of $C_0(Q)$ that has the property* awc_1, *and $x_1 < x_2 < \cdots < x_k$, $1 \leqslant k \leqslant n$, be an N-totally dependent subset of Q. If Q contains at least $n+2$ points, then either $[x_1, x_k] = \{x_1, x_2, ..., x_k\}$ or there is i_0 in $\{1, 2, ..., k-1\}$ such that $Q \backslash (x_{i_0}, x_{i_0+1}) = \{x_1, x_2, ..., x_k\}$.*

Proof. If $k = 1$ then there is nothing to prove, so one can assume that $k \geqslant 2$ and that $[x_1, x_k] \neq \{x_1, x_2, ..., x_k\}$. It will be shown that there is $i_0 \in \{1, 2, ..., k-1\}$ such that $Q \backslash (x_{i_0}, x_{i_0+1}) = \{x_1, x_2, ..., x_k\}$.

Since $[x_1, x_k] \neq \{x_1, x_2, ..., x_k\}$ it follows that there is i_0 in $\{1, 2, ..., k-1\}$ such that the open interval (x_{i_0}, x_{i_0+1}) is not empty. It will be shown that $Q \backslash (x_{i_0}, x_{i_0+1}) = \{x_1, x_2, ..., x_k\}$; that is, the set

$$A = \{x \in Q; x \leqslant x_{i_0} \text{ or } x \geqslant x_{i_0+1}\} \backslash \{x_1, x_2, ..., x_k\}$$

is empty.

Assume not, and let $y_0 \in A$. Since $(x_{i_0}, x_{i_0+1}) \neq \emptyset$, let $x_0 \in (x_{i_0}, x_{i_0+1})$. Either $y_0 > x_{i_0+1}$ or $y_0 < x_{i_0}$. The proof will be given for the case when $y_0 > x_{i_0+1}$; the proof for the other case is similar.

Since $\{x_1, x_2, ..., x_k\}$ is N-totally dependent, there are nonzero real numbers $\lambda, \lambda_1, ..., \lambda_{i_0-1}, \lambda_{i_0+2}, ..., \lambda_k$ such that

$$x_{i_0+1} = \lambda x_{i_0} + \sum_{\substack{i=1 \\ i \notin \{i_0, i_0+1\}}} \lambda_i x_i \tag{*}$$

Let $t_1 < t_2 < \cdots < t_n$ be a subset of Q satisfying the following properties:

(a) $\{x_1, x_2, ..., x_{i_0-1}, x_0, x_{i_0+1}, ..., x_k\} \subseteq \{t_1, t_2, ..., t_n\}$,

(b) $x_{i_0} \notin \{t_1, t_2, ..., t_n\}$ and $y_0 \notin \{t_1, t_2, ..., t_n\}$.

This can be done because $k \leqslant n$ and Q contains at least $(n+2)$ points. By defining $t_0 = -\infty$ and $t_{n+1} = +\infty$, one can find j_0 in $\{1, 2, ..., n+1\}$ such that $x_{i_0} \in (t_{j_0-1}, t_{j_0})$. Also there is $m > 1$ such that $x_{i_0+1} = t_m$. Let $z_1 < z_2 < \cdots < z_n$ be the set obtained from the set $\{t_1, t_2, ..., t_n\}$ by replacing x_0 by y_0. Then since $x_{i_0} < x_0 < x_{i_0+1}$ and $y_0 > x_{i_0+1}$, it follows that $x_{i_0} \in (z_{j_0-1}, z_{j_0})$ and $x_{i_0+1} = z_{m-1}$. By Lemma 2.1 the points $t_1, ..., t_n$ are N-independent, and the points $z_1, ..., z_n$ are N-independent. Thus if $\{g_1, g_2, ..., g_n\}$ is any basis for N, it follows that

$$|t_1, t_2, ..., t_n| \neq 0 \qquad \text{and} \qquad |z_1, z_2, ..., z_n| \neq 0.$$

But N is a weak Chebyshev subspace of $C_0(Q)$, so

$$|t_1, t_2, ..., t_m, ..., t_n| \cdot |z_1, z_2, ..., z_{m-1}, ..., z_n| > 0.$$

By (*),

$$\begin{aligned} &|t_1, ..., t_{j_0-1}, t_{j_0}, ..., t_m, ..., t_n| \\ &\quad = \lambda\, |t_1, ..., t_{j_0-1}, t_{j_0}, ..., t_{m-1}, x_{i_0}, t_{m+1}, ..., t_n| \\ &\quad = \lambda(-1)^{m-j_0}\, |t_1, ..., t_{j_0-1}, x_{i_0}, t_{j_0}, ..., t_{m-1}, t_{m+1}, ..., t_n|. \end{aligned}$$

Also

$$\begin{aligned} &|z_1, ..., z_{j_0-1}, z_{j_0}, ..., z_{m-1}, ..., z_n| \\ &\quad = \lambda\, |z_1, ..., z_{j_0-1}, z_{j_0}, ..., z_{m-2}, x_{i_0}, z_m, ..., z_n| \\ &\quad = \lambda(-1)^{m-j_0-1}\, |z_1, ..., z_{j_0-1}, x_{i_0}, z_{j_0}, ..., z_{m-2}, z_m, ..., z_n|. \end{aligned}$$

Thus

$$\begin{aligned} -\lambda^2\, &|t_1, ..., t_{j_0-1}, x_{i_0}, t_{j_0}, ..., t_{m-1}, t_{m+1}, ..., t_n| \\ &\cdot |z_1, ..., z_{j_0-1}, x_{i_0}, z_{j_0}, ..., z_{m-2}, z_m, ..., z_n| > 0. \end{aligned}$$

But $t_1 < t_2 < \cdots < t_{j_0-1} < x_{i_0} < t_{j_0} < \cdots < t_n$, $z_1 < z_2 < \cdots < z_{j_0-1} < x_{i_0} < z_{j_0} < \cdots < z_n$, and N is a weak Chebyshev subspace. Therefore

$$|t_1, ..., t_{j_0-1}, x_{i_0}, t_{j_0}, ..., t_{m-1}, t_{m+1}, ..., t_n| \cdot |z_1, ..., z_{j_0-1}, x_{i_0}, z_{j_0}, ..., z_{m-2}, z_m, ..., z_n| \geqslant 0$$

so $-\lambda^2 > 0$, which is a contradiction.

The following lemma will be used frequently in Section 3.

2.4. LEMMA. *Let Q be a locally compact totally ordered space, let N be an n-dimensional subspace of $C_0(Q)$, and let $x_1 < x_2 < \cdots < x_{n+1}$ be $(n+1)$ points of Q. Assume that there is $g \neq 0$ in N such that g oscillates weakly at the points $x_1, x_2, ..., x_{n+1}$. Let $\{x_{i_1}, x_{i_2}, ..., x_{i_k}\}$ be the set of all points in $\{x_1, ..., x_{n+1}\}$ at which $g \equiv 0$. If the set $\{x_{i_1}, ..., x_{i_k}\}$ is empty or N-independent, then N is not a weak Chebyshev subspace of $C_0(Q)$.*

Proof. If the set $\{x_{i_1}, ..., x_{i_k}\}$ is empty, then g oscillates at $(n+1)$ points of Q. By Theorem 1.5, N is not a weak Chebyshev subspace of $C_0(Q)$. Now assume that the set $\{x_{i_1}, ..., x_{i_k}\}$ is a nonempty N-independent subset of Q. Then $1 \leqslant k \leqslant n$. Since g oscillates weakly at $x_1 < x_2 < \cdots < x_{n+1}$, one may assume that

$$(-1)^i g(x_i) \geqslant 0 \qquad \text{for} \quad i = 1, 2, ..., n+1.$$

Let $\lambda = \frac{1}{2} \min\{|g(x_i)|;\ g(x_i) \neq 0,\ i = 1, 2, ..., n+1\}$. Then $\lambda > 0$. By Lemma 1.3, there is g' in N such that

$$g'(x_{i_j}) = (-1)^{i_j} \qquad \text{for} \quad j = 1, 2, ..., k.$$

Let $h = g + \lambda(g'/\|g'\|)$. Then $h \in N$ and h oscillates at $x_1 < x_2 < \cdots < x_{n+1}$. Thus by Theorem 1.5, N is not a weak Chebyshev subspace.

3. THE EQUIVALENCE BETWEEN awc_1 AND awc_2

In this section it is shown that if Q is a locally compact totally ordered space, and N is an n-dimensional weak Chebyshev subspace of $C_0(Q)$, then N has the property awc_1 if and only if it has the property awc_2. Therefore, by Theorem 1.1, if the weak Chebyshev subspace N has the property awc_1, it follows that the metric projection P_N has a continuous selection. Combining this result with a result of Brown [1], it is shown also that if the n-dimensional weak Chebyshev subspace N is a Z-subspace, then P_N has a continuous selection if and only if N has the property awc_1.

3.1. THEOREM. *Let Q be a locally compact totally ordered space that contains at least $(n+1)$ points, and let N be an n-dimensional weak Chebyshev subspace of $C_0(Q)$. If N has the property* awc_1, *then it has the property* awc_2.

Proof. Let $f \in C_0(Q)$. By Theorem 1.5, there is $g \in N$ such that $d(f, N) = \|f-g\|$, and $(f-g)$ equioscillates at $(n+1)$ points. It will be shown that g is unique.

If $f \in N$ then there is nothing to prove. So assume that $f \notin N$, and that there is another g' in N such that $d(f, N) = \|f-g'\|$, and $(f-g')$ equioscillates at $(n+1)$ points. Without loss of generality one may assume that $g \neq 0$ and $g' = 0$.

Since $(f-g)$ and f equioscillate at $(n+1)$ points, it follows that there are $x_1 < x_2 < \cdots < x_{n+1}$ in Q, $y_1 < y_2 < \cdots < y_{n+1}$ in Q, and $\varepsilon_1 = \pm 1$, $\varepsilon_2 = \pm 1$, such that

$$(-1)^i (f-g)(x_i) = \varepsilon_1 \|f-g\| = \varepsilon_1 d(f, N), \qquad i = 1, 2, ..., n+1$$
$$(-1)^i f(y_i) = \varepsilon_2 \|f\| = \varepsilon_2 d(f, N), \qquad i = 1, 2, ..., n+1.$$

Thus for each $i = 1, 2, ..., n+1$, one has

$$\begin{aligned}(-1)^i \varepsilon_1 g(x_i) &= (-1)^i \varepsilon_1 f(x_i) - (-1)^i \varepsilon_1 (f-g)(x_i) \\ &= (-1)^i \varepsilon_1 f(x_i) - d(f, N) \leqslant 0\end{aligned}$$

and

$$\begin{aligned}(-1)^i \varepsilon_2 g(y_i) &= (-1)^i \varepsilon_2 f(y_i) - (-1)^i \varepsilon_2 (f-g)(y_i) \\ &= d(f, N) - (-1)^i \varepsilon_2 (f-g)(y_i) \geqslant 0.\end{aligned}$$

That is, g oscillates weakly at $x_1 < x_2 < \cdots < x_{n+1}$ and at $y_1 < y_2 < \cdots < y_{n+1}$.

Let $\{t_1, ..., t_\iota\}$ be the set of all zeros of g in $\{x_1, ..., x_{n+1}, y_1, ..., y_{n+1}\}$.

Since N has the property awc_1, it follows that $\iota \leqslant n$. If $\{t_1, ..., t_\iota\}$ is empty or N-independent, then by applying Lemma 2.4 to the set $\{x_1, x_2, ..., x_{n+1}\}$ or to the set $\{y_1, y_2, ..., y_{n+1}\}$ one can conclude that N is not a weak Chebyshev subspace. So one may assume that $\iota \geqslant 1$ and that the set $\{t_1, ..., t_\iota\}$ is N-dependent. Let $\{z_1, z_2, ..., z_k\}$ be a nonempty N-totally dependent subset of $\{t_1, ..., t_\iota\}$. Then $k \leqslant n$. It will be shown that

$$\{z_1, z_2, ..., z_k\} \subseteq \{x_1, x_2, ..., x_{n+1}\} \cap \{y_1, y_2, ..., y_{n+1}\}.$$

Assume not. Then there is i_0 such that $z_{i_0} \notin \{x_1, ..., x_{n+1}\}$ or $z_{i_0} \notin \{y_1, ..., y_{n+1}\}$. By Lemma 2.1 the set $\{t_1, t_2, ..., t_\iota\} \setminus \{z_{i_0}\}$ is N-independent. Thus if $z_{i_0} \notin \{x_1, x_2, ..., x_{n+1}\}$, then the points of the set $\{t_1, ..., t_\iota\} \cap$

$\{x_1, x_2, ..., x_{n+1}\}$ are N-independent. By applying Lemma 2.4 to the set $\{x_1, x_2, ..., x_{n+1}\}$ and the function g, one can conclude that N is not a weak Chebyshev subspace of $C_0(Q)$, which is a contradiction. The same contradiction can be obtained if $z_{i_0} \notin \{y_1, y_2, ..., y_{n+1}\}$.

Since $\{z_1, z_2, ..., z_k\} \subseteq \{x_1, x_2, ..., x_{n+1}\} \cap \{y_1, y_2, ..., y_{n+1}\}$, it follows that there are i_0 and j_0 in $\{1, 2, ..., n+1\}$ such that $z_1 = x_{i_0} = y_{j_0}$. Without loss of generality one might consider the following two cases only.

Case 1. $i_0 = j_0$.
In this case $\varepsilon_1 = \varepsilon_2$ since otherwise

$$\begin{aligned}(-1)^{i_0}\varepsilon_1 g(x_{i_0}) &= (-1)^{i_0}\varepsilon_1 f(x_{i_0}) - (-1)^{i_0}\varepsilon_1 (f-g)(x_{i_0})\\ &= -(-1)^{i_0}\varepsilon_2 f(y_{i_0}) - (-1)^{i_0}\varepsilon_1 (f-g)(x_{i_0})\\ &= -2d(f, N) \neq 0.\end{aligned}$$

Also $(-1)^i \varepsilon_1 g(x_i) \leqslant 0$ and $(-1)^i \varepsilon_1 g(y_i) \geqslant 0$ for each $i = 1, 2, ..., n+1$, so whenever $x_i = y_i$ one has $g(x_i) = 0$. If $x_i = y_i$ for each $i = 1, 2, ..., n+1$, then g has at least $(n+1)$ zeros, which contradicts the fact that N has the property awc_1. Thus there is m_0 such that $x_{m_0} \neq y_{m_0}$. Without loss of generality assume that $x_{m_0} < y_{m_0}$. Obviously $i_0 \neq m_0$, so either $i_0 < m_0$ or $i_0 > m_0$. If $i_0 < m_0$, then since $g(y_{i_0-1}) \cdot g(x_{i_0+1}) \leqslant 0$ and $g(x_{m_0}) \cdot g(y_{m_0}) \leqslant 0$, it follows that g oscillates weakly at the $(n+1)$ points

$$y_1 < y_2 < \cdots < y_{i_0-1} < x_{i_0+1} < \cdots < x_{m_0} < y_{m_0} < \cdots < y_{n+1}.$$

Since $z_1 = x_{i_0} \notin \{y_1, ..., y_{i_0-1}, x_{i_0+1}, ..., x_{m_0}, y_{m_0}, ..., y_{n+1}\}$, it follows by Lemma 2.1 that the set

$$\{t_1, t_2, ..., t_l\} \cap \{y_1, ..., y_{i_0-1}, x_{i_0+1}, ..., x_{m_0}, ..., y_{n+1}\}$$

is N-independent. Thus by applying Lemma 2.4 to the set $\{y_1, ..., y_{i_0-1}, x_{i_0+1}, ..., x_{m_0}, y_{m_0}, ..., y_{n+1}\}$ and the function g, one can conclude that N is not a weak Chebyshev subspace, which is a contradiction. If $m_0 < i_0$, then, by applying Lemma 2.4 to the set

$$\{x_1, x_2, ..., x_{m_0}, y_{m_0}, ..., y_{i_0-1}, x_{i_0+1}, ..., x_{n+1}\}$$

and the function g, one can conclude that N is not a weak Chebyshev subspace.

Case 2. $i_0 < j_0$.
If $\varepsilon_1 = -\varepsilon_2$, then $(-1)^i \varepsilon_2 g(x_i) \geqslant 0$ and $(-1)^i \varepsilon_2 g(y_i) \geqslant 0$ for each $i = 1, 2, ..., n+1$. Also since $i_0 < j_0$, it follows that $y_{i_0} < y_{j_0} = x_{i_0} < x_{i_0+1}$. Therefore g oscillates weakly at the $(n+1)$ points

$$y_1 < y_2, ..., y_{i_0} < x_{i_0+1} < \cdots < x_{n+1}.$$

Since $y_{i_0} < y_{j_0} = x_{i_0} < x_{i_0+1}$ it follows that $z_1 = x_{i_0}$ is not an element in the set $\{y_1, y_2, ..., y_{i_0}, x_{i_0+1}\}$. Therefore, by Lemma 2.1, the set $\{t_1, ..., t_\iota\} \cap \{y_1, ..., y_{i_0}, x_{i_0+1}, ..., x_{n+1}\}$ is N-independent. Thus it follows by Lemma 2.4 that N is not a weak Chebyshev subspace. This is a contradiction.

If $\varepsilon_1 = \varepsilon_2$, then for each $i = 1, 2, ..., n+1$),

$$(-1)^i \varepsilon_1 g(x_i) \leqslant 0 \text{ and } (-1)^i \varepsilon_1 g(y_i) \geqslant 0.$$

It will be shown that $i_0 < j_0 - 1$. Assume not. Then $i_0 = j_0 - 1$, so $x_{i_0} = y_{j_0} = y_{i_0+1}$. But then

$$\begin{aligned}(-1)^{i_0} \varepsilon_1 g(x_{i_0}) &= (-1)^{i_0} \varepsilon_1 f(x_{i_0}) - (-1)^{i_0} \varepsilon_1 (f-g)(x_{i_0}) \\ &= -(-1)^{i_0+1} \varepsilon_2 f(y_{i_0+1}) - (-1)^{i_0} \varepsilon_1 (f-g)(x_{i_0}) \\ &= 2d(f, N) \neq 0.\end{aligned}$$

Since $i_0 < j_0 - 1$, it follows that $y_{i_0-1} < y_{j_0} = x_{i_0} < x_{j_0-1}$. Thus the point $z_1 = x_{i_0}$ is not any of the $(n+1)$ points

$$y_1 < y_2 < \cdots < y_{j_0-1} < x_{j_0-1} < \cdots < x_n.$$

Therefore the set $\{t_1, ..., t_\iota\} \cap \{y_1, ..., y_{j_0-1}, x_{j_0-1}, ..., x_n\}$ is N-independent, but g oscillates weakly at the points $y_1 < y_2 < \cdots < y_{j_0-1} < x_{j_0-1} < \cdots < x_n$. Thus by Lemma 2.4, the subspace N is not weak Chebyshev, which is a contradiction.

3.2. LEMMA. *Let Q be a locally compact totally ordered space that contains at least $(n+1)$ points, and let N be an n-dimensional weak Chebyshev subspace of $C_0(Q)$. If N has the property* awc$_2$, *then it has the property* awc$_1$.

Proof. Assume that N does not have the property awc$_1$. Then there is $g \in N$ such that $\|g\| = 1$ and g has at least $(n+1)$ zeros. It will be shown that there is $f \in C_0(Q)$ such that g and 0 are best approximations for f from N and $(f-g)$ and $(f-0)$ equioscillate at $(n+1)$ points.

Let $x_1 < x_2 < \cdots < x_{n+1}$ be $(n+1)$ zeros of g. Since Q is a locally compact totally ordered space, it follows that there are functions h and h' in $C_0(Q)$ satisfying the following properties:

(a) $0 \leqslant h(x) \leqslant 1$ and $0 \leqslant h'(x) \leqslant 1$ for each $x \in Q$

(b) $h(x_i) = 1$ for each $i = 1, 2, ..., n+1$, and

$$h'(x_i) = \begin{cases} 1 & \text{if } i \text{ is even} \\ 0 & \text{if } i \text{ is odd.} \end{cases}$$

Define

$$f_1(x)=\begin{cases} h(x), & \text{if} \quad g(x)\geqslant 0,\\ h(x)+g(x), & \text{if} \quad g(x)<0,\end{cases}$$

$$f_2(x)=\begin{cases} -h(x)+g(x), & \text{if} \quad g(x)>0\\ -h(x); & \text{if} \quad g(x)\leqslant 0.\end{cases}$$

Then f_1 and f_2 are elements in $C_0(Q)$. Furthermore one can easily show that $\|f_1\|=1$, $\|f_2\|=1$, $\|f_1-g\|=1$, and $\|f_2-g\|=1$.

Let $f(x)=h'(x)\,f_1(x)+(1-h')(x)\,f_2(x)$. Then since $0\leqslant h'(x)\leqslant 1$, it follows that $\|f\|\leqslant 1$ and $\|f-g\|\leqslant 1$. Now for each $i\in\{1, 2, ..., n+1\}$, if i is even, then

$$\begin{aligned}(f-g)(x_i)=f(x_i)&=h'(x_i)\,f_1(x_i)+(1-h')(x_i)\,f_2(x_i)\\ &=f_1(x_i)=h(x_i)=1,\end{aligned}$$

and if i is odd, then

$$\begin{aligned}(f-g)(x_i)=f(x_i)&=h'(x_i)\,f_1(x_i)+(1-h')(x_i)\,f_2(x_i)\\ &=f_2(x_i)=\ -h(x_i)=\ -1.\end{aligned}$$

So $\|f-g\|=1$, and $(f-g)$ equioscillates at $(n+1)$ points of Q. Therefore by [5, Lemma 2.3], it follows that g is a best approximation for f from N. On the other hand, $\|f-0\|=\|f\|=1$ so 0 is another best approximation for f from N, and since $f(x_i)=(-1)^i$ for each $1=1, 2, ..., n+1$, it follows that $f-0$ equioscillates at $(n+1)$ points.

3.3. THEOREM. *Let Q be a locally compact totally ordered space that contains at least $(n+1)$ points, and let N be an n-dimensional weak Chebyshev subspace of $C_0(Q)$. Then N has the property* awc$_1$ *if and only if it has the property* awc$_2$.

Proof. It follows from Theorem 3.1 and Lemma 3.2.

3.4. THEOREM. *Let Q be a locally compact totally ordered spoace, and let N be an n-dimensional subspace of $C_0(Q)$. If N is a weak Chebyshev subspace and has the property* awc$_1$, *then the metric projection P_N has a continuous selection.*

Proof. It follows from Theorem 1.1 and Theorem 3.3.

In the case when N is a finite dimensional Z-subspace of $C[a, b]$, where $[a, b]$ is a compact real interval, Nurnberger [7] showed that the existence of a continuous selection for P_N is equivalent to the fact that N

is a weak Chebyshev subspace and has the property awc_1. However, in general this is not true. The following example shows that if $Q=[-2,-1]\cup[1,2]$, then for each $n\geqslant 1$, there is an n-dimensional Z-subspace N of $C(Q)$ such that N is not a weak Chebyshev subspace, and the metric projection P_N has a continuous selection.

3.5. EXAMPLE. Let $Q=[-2,-1]\cup[1,2]$ and, for each $n\geqslant 1$, let N be the n-dimensional subspace of $C(Q)$ generated by the polynomials $\{x, x^2, ..., x^n\}$. Then each $g\neq 0$ in N has at most $(n-1)$ zeros in Q, so N is a Chebyshev subspace of $C(Q)$. Thus by Haar's theorem (see Singer [10, Theorem 2.2, p. 215]), for each $f\in C(Q)$, the set $P_N(f)$ is a singleton. But then it is well known and easy to show that $P_N\colon C(Q)\to N$ is continuous. On the other hand, if $g_i(x)=x^i$ for each $1\leqslant i\leqslant n$, then one can find $x_1<x_2<\cdots<x_n$ and $y_1<y_2<\cdots<y_n$ in Q such that

$$\det[g_i(x_j)]\det[g_i(y_j)]<0.$$

Thus N is not a weak Chebyshev subspace.

3.6. THEOREM. *Let Q be a compact totally ordered space and let N be an n-dimensional weak Chebyshev Z-subspace of $C(Q)$. Then the metric projection P_N has a continuous selection if and only if N has the property* awc_1.

Proof. This follows from Theorem 1.1, Theorem 3.4, and Brown [1, the corollary of Lemma 2].

In theorem 3.6 the fact that N is a Z-subspace of $C(Q)$ is essential. The following example shows that when N is not a Z-subspace, then Theorem 3.6 need not be true.

3.7. EXAMPLE. Let $n\geqslant 2$ be given. For each $1\leqslant k\leqslant n$ let $I_k=[k-\frac{1}{4}, k+\frac{1}{4}]$ and let $Q=\bigcup_{k=1}^{n} I_k$. For each $1\leqslant k\leqslant n$, define $g_k\in C(Q)$ as follows:

$$g_k(x)=\begin{cases}1 & \text{if } x\in I_k\\ 0 & \text{otherwise.}\end{cases}$$

Let N be the n-dimensional subspace of $C(Q)$ generated by $\{g_1, g_2, ..., g_n\}$, then N is not a Z-subspace of $C(Q)$ and does not have the property awc_1. In order to show that N is a weak Chebyshev subspace, it is enough to note that for each $x_1<x_2<\cdots<x_n$ in Q, it is always true that

$$\det[g_k(x_i)]\geqslant 0.$$

Furthermore if $N_k = N|_{I_k} = \{g|_{I_k};\ g \in N\}$, then N_k is a one-dimensional Chebyshev subspace of $C(I_k)$. It will be shown that the metric projection $P_N: C(Q) \to 2^N$ has a continuous selection. Let $f \in C(Q)$, and let $f_k = f|_{I_k}$. Then there is a unique real number $\alpha_k(f)$ in R such that $\alpha_k(f)\, g_k$ is the best approximation of f_k from N_k. If $g_f = \sum_{k=1}^{n} \alpha_k(f)\, g_k$, then g_f is a best approximation for f from N.

Define $\psi: C(Q) \to N$ by $\psi(f) = g_f$. Then $\psi(f) \in P_N(f)$ for each $f \in C(Q)$. Furthermore, if $\{f^i\}$ is a sequence in $C(Q)$ that converges to f_0, then for each $1 \leqslant k \leqslant n$, the sequence $\{f_k^i\}$ converges to f_k^0. Since N_k is a Chebyshev subspace, it follows that the sequence $\{\alpha_k(f^i)\, g_k\}$ converges to $\alpha_k(f^0)\, g_k$. Thus the sequence $\{\psi(f^i)\}$ converges to $\psi(f^0)$. That is, ψ is a continuous selection for P_N.

4. An Extension of Mairhuber's Theorem

In this section it will be shown that if Q is a compact totally ordered space, and $C(Q)$ contains an n-dimensional weak Chebyshev subspace that has the property awc_1, where $n \geqslant 3$, then Q is homeomorphic to a subset of the real numbers R. This result together with the results of Section 3 shows that if Q is a compact totally ordered space and $C(Q)$ contains an n-dimensional weak Chebyshev Z-subspace, where $n \geqslant 3$, such that the metric projection P_N has a continuous selection, then Q is homeomorphic to a subset of the real numbers. The case when $n = 2$ is discussed also, and an example will be given to show that there is a compact totally ordered space Q, that is not homeomorphic to any subset of R, such that $C(Q)$ contains a 2-dimensional weak Chebyshev Z subspace N for which the metric projection P_N has a continuous selection.

This result is similar to Mairhuber's theorem. Mairhuber's theorem asserts that if Q is a compact Hausdorff space, and $C(Q)$ contains a finite-dimensional Chebyshev subspace of dimension more than one, then Q is homeomorphic to a subset of a circle (Schoenberg and Yang [9]). Every proper compact subset of the circle is homeomorphic to a subset of the real numbers. In this paper only totally ordered spaces are considered, and since the topology of the circle cannot be defined by a totally ordering relation, it follows that any compact totally ordered subspace of the circle must be a proper subspace. Thus Mairhuber's theorem can be restated as follows.

4.1. Theorem (Mairhuber's Theorem). *Let Q be a compact totally ordered space. If $C(Q)$ contains a finite-dimensional Chebyshev subspace of dimension not less than two, then Q is homeomorphic to a subset of the real numbers.*

Schoenberg and Yang [9] proved that if Q is a compact Hausdorff space with the property that for any nonempty open subset $U \subseteq Q$, the set $Q \backslash U$ is homeomorphic to a subset of the circle S^1, then either Q is homeomorphic to a subset of the circle, or Q is homeomorphic to the union $S^1 \cup \{a\}$, where a is a point outside S^1. This result can be restated in the case when Q is totally ordered as follows.

4.2. PROPOSITION. *Let Q be a compact totally ordered space. If for each nonempty open subset $U \subseteq Q$, the set $Q \backslash U$ is homeomorphic to a subset of the real numbers, then Q is homeomorphic to a subset of the real numbers.*

The proof of the following proposition is elementary.

4.3. PROPOSITION. *Let Q be a locally compact Hausdorff space, let N be an n-dimensional subspace of $C_0(Q)$ that has the property* awc_1, *and let A be any closed subset of Q. If $N|_A$ is of dimension less than n, then A consists of a finite number of points.*

4.4. LEMMA. *Let Q be a compact totally ordered space, and let N be a two-dimensional weak Chebyshev subspace that has the property* awc_1. *If there is $x_0 \in Q$ such that $g(x_0) = 0$ for each $g \in N$, then Q is homeomorphic to a subset of the real numbers R.*

Proof. If x_0 is an isolated point of Q, then $Q' = Q \backslash \{x_0\}$ is compact and each $g \neq 0$ in the 2-dimensional subspace $N' = N|_{Q'}$ has at most one zero. By Theorem 4.1, the set Q' is homeomorphic to a subset of the real numbers. But then Q is also homeomorphic to a subset of the real numbers. Assume that x_0 is a limit point in Q, and let $Q_1 = \{x \in Q; x < x_0\}$, $Q_2 = \{x \in Q; x > x_0\}$. If x_0 is not a limit point for Q_1, then Q_1 is compact, so by either Theorem 4.1 or Proposition 4.3, the set Q_1 is homeomorphic to a subset of the real numbers. Thus to prove this lemma, it is enough to show that $\{x \in Q; x > x_0\}$ is homeomorphic to a subset of the real numbers. The same argument is true if x_0 is not a limit point for Q_2. Therefore the proof will be given only for the following two cases:

Case 1. x_0 is a limit point for both Q_1 and Q_2.

Let U be a nonempty open subset of Q. By Proposition 4.2 it is enough to show that $Q' = Q \backslash U$ is homeomorphic to a subset of R. If $\dim N|_{Q'} < 2$ or $x_0 \notin Q'$, then by either Proposition 4.3 or Theorem 4.1, the set Q' is homeomorphic to a subset of R. Thus one may assume that $x_0 \in Q'$, and $\dim N|_{Q'} = 2$.

Let $q_0 \in U$. Then either $x_0 < q_0$ or $x_0 > q_0$. Without loss of generality, assume that $x_0 > q_0$. Then Q' is the union of the two disjoint compact sets

$Q_0 = \{x \in Q'; x > q_0\}$ and $Q_0' = \{x \in Q'; x < q_0\}$. Since $x_0 \notin Q_0'$, it follows by either Proposition 4.3 or Theorem 4.1 that Q_0' is homeomorphic to a subset of R. Thus it is enough to show that $Q_0 = \{x \in Q'; x > q_0\}$ is homeomorphic to a subset of R.

Since N is a two-dimensional weak Chebyshev subspace of $C(Q)$, it follows by Theorem 1.5 that there is a basis $\{g_1, g_2\}$ of N, and $\varepsilon = \pm 1$, such that $g_2(q_0) = 0$, $g_2(x) \leqslant 0$ for $x \leqslant q_0$, $g_2(x) \geqslant 0$ for $x \geqslant q_0$, and for each $x < y$ in Q, it is always true that

$$\varepsilon[g_1(x)\, g_2(y) - g_2(x)\, g_1(y)] \geqslant 0.$$

Without loss of generality assume that $\varepsilon = 1$. Then since each $g \neq 0$ in N has at most one zero in $Q \backslash \{x_0\}$, it follows that for each $x < y$ in $Q \backslash \{x_0\}$ it is always true that

$$[g_1(x)\, g_2(y) - g_2(x)\, g_1(y)] > 0.$$

Therefore, since $g_2(x) > 0$ for each $x \in Q_0 \backslash \{x_0\}$, it follows that the function $h(x) = (g_1(x)/g_2(x))$ is a continuous, strictly decreasing real-valued function on $Q_0 \backslash \{x_0\}$. Furthermore, if x_1 and x_2 are two points in Q such that $q_0 < x_1 < x_0 < x_2$, then for each $x \neq x_0$ in the interval $[x_1, x_2]$ one always has

$$h(x_1) > h(x) > h(x_2).$$

Thus $\lim_{x \to x_0,\, x < x_0} h(x) = a$ exists and is finite, and $\lim_{x \to x_0,\, x > x_0} = b$ exists and is finite. Also if $q_0 < x < x_0 < y$ in Q, then

$$h(x) > a \geqslant b > h(y).$$

Define $\psi\colon Q_0 \to R$ as follows:

$$\psi(x) = \begin{cases} h(x) - a + b & \text{if } \; x < x_0 \\ b & \text{if } \; x = x_0 \\ h(x) & \text{if } \; x > x_0. \end{cases}$$

Then ψ is a continuous strictly increasing function from the compact space Q_0 onto the Hausdorff space $\psi(Q_0) \subseteq R$. Thus it is a homeomorphism; that is, Q_0 is homeomorphic to a subset of R.

Case 2. $Q_1 = \varnothing$ or $Q_2 = \varnothing$.

Without loss of generality assume that $Q_1 = \varnothing$; that is, $Q = \{x \in Q;\; x \geqslant x_0\}$, and x_0 is a limit point for Q. Since Q is compact, it follows that there is q_0 in Q such that $q_0 \geqslant x$ for each $x \in Q$. If q_0 is an isolated point for Q, then it is enough to show that $Q \backslash \{q_0\}$ is homeomorphic to a subset of R. Let U be a nonempty open subset of Q if q_0 is a limit point, and let U be a nonempty open subset of $Q\backslash\{q_0\}$ if q_0 is an isolated point of Q. Also let $q_1 \neq q_0$ be an element in U. As in Case 1, it is enough to show that the compact set $Q_0 = \{x \in Q\backslash U; x < q_1\}$ is homeomorphic to a subset of R. If $x_0 \notin Q_0$, then there is nothing to prove. Thus one may assume that $x_0 \in Q_0$. As in Case 1, one can find a basis $\{g_1, g_2\}$ for N such that $g_1(q_1) = 0$, $g_1(x) \geqslant 0$ for $x \leqslant q_1$, $g_2(q_0) = 0$, and $g_2(x) \geqslant 0$ for $x \leqslant q_0$. Since N has the property awc$_1$, it follows that both g_1 and g_2 are positive functions on $Q_0\backslash\{x_0\}$. Thus, as in Case 1, there exists $h \in \{g_1/g_2, g_2/g_1\}$ such that h is a strictly increasing continuous positive-valued function from $Q_0\backslash\{x_0\}$ into R. Let $a = \lim_{x \to x_0} h(x)$. Then $0 \leqslant a < \infty$, and define $\psi: Q_0 \to R$ by

$$\psi(x) = \begin{cases} h(x) & \text{if } x \neq x_0 \\ a & \text{if } x = x_0. \end{cases}$$

Then, as in Case 1, one can show that ψ is a homeomorphism from Q_0 onto a subset of R.

4.5. Lemma. *Let Q be a compact totally ordered space, and let N be an n-dimensional weak Chebyshev subspace of $C(Q)$ that has the property* awc$_1$. *If $n \geqslant 2$ and there is $x_0 \in Q$ such that $g(x_0) = 0$ for each $g \in N$, then Q is homeomorphic to a subset of R.*

Proof. By induction. If $n = 2$, then by Lemma 4.4, the hypothesis is true. Assume that the hypothesis is true for $n - 1 \geqslant 2$. It will be shown that it is true for n.

Let U be a nonempty open subset of Q. By Proposition 4.2, it is enough to show that $Q' = Q\backslash U$ is homeomorphic to a subset of R. If $\dim N|_{Q'} < n$ or $x_0 \in U$, then by either Proposition 4.3 or Theorem 4.1, the set Q' is homeomorphic to a subset of R. Thus one may assume that $\dim N|_{Q'} = n$ and $x_0 \in Q'$. Let $q_0 \in U$, $Q_1 = \{x \in Q'; x < q_0\}$, and $Q_2 = \{x \in Q'; x > q_0\}$. Then Q' is the union of the two disjoint compact sets Q_1 and Q_2. Without loss of generality assume that $x_0 \in Q_2$. Therefore, by either Proposition 4.3 or Theorem 4.1, the set Q_1 is homeomorphic to a subset of R. Thus it is enough to show that Q_2 is homeomorphic to a subset of R. If $\dim N|_{Q_2} < n$ then by Proposition 4.3, the set Q_2 is homeomorphic to a subset of R, so one may assume that $\dim N|_{Q_2} = n$. Since N is a weak Chebyshev subspace of $C(Q)$ that has the property awc$_1$, it follows that there is a basis $\{g_0, g_1, ..., g_{n-1}\}$ of N and $\varepsilon = \pm 1$, such that $g_0(q_0) = 1$, $g_i(q_0) = 0$ for each $i \geqslant 1$, and for each $q_1 < q_2 < ..., q_{n-1}$ in Q_0,

$$\varepsilon \det[g_i(q_j)]_{i=0, j=0}^{n-1, n-1} = \varepsilon \det[g_i(q_j)]_{i=1, j=1}^{n-1, n-1} \geqslant 0,$$

and equal to zero if and only if $x_0 \in \{q_1, ..., q_{n-1}\}$. Let N' be the $(n-1)$-dimensional subspace of $C(Q_0)$ generated by the restriction of $\{g_1, g_2, ..., g_{n-1}\}$ on Q_0, then N' is a weak Chebyshev subspace of $C(Q_0)$, that has the property awc_1, and $g(x_0)=0$ for each $g \in N'$. Thus since the hypothesis is true for $(n-1)$, it follows that Q_0 is homeomorphic to a subset of R.

4.6. LEMMA. *Let Q be a locally compact totally ordered space, let N be an n-dimensional weak Chebyshev subspace of $C(Q)$ that has the property* awc_1, *and assume that $n \geqslant 3$. If there are $x_1 < x_2$ in Q such that the set $\{x_1, x_2\}$ is N-totally dependent, then Q is homeomorphic to a subset of R.*

Proof. If Q is finite then Q is homeomorphic to a subset of R. Otherwise by Theorem 2.3, either $Q = \{x \in Q; X \leqslant x_1\} \cup \{x \in Q; x \geqslant x_2\}$ or $Q = \{x \in Q; x_1 \leqslant x \leqslant x_2\}$. If $Q = \{x \in Q; x \leqslant x_1\} \cup \{x \in Q; x \geqslant x_2\}$, then let $Q_1 = \{x \in Q; x \leqslant x_1\}$ and $Q_2 = \{x \in Q; x \geqslant x_2\}$. If $Q = \{x \in Q; x_1 \leqslant x \leqslant x_2\}$, then let U be any nonempty open subset of Q. By Proposition 4.2 it is enough to show that $Q \backslash U$ is homeomorphic to a subset of R. Let $q_0 \in U$ and let $Q_1 = \{x \in Q \backslash U; x < q_0\}$, $Q_2 = \{x \in Q \backslash U; x \geqslant q_0\}$. Then $Q \backslash U$ is the union of the two disjoint compact sets Q_1 and Q_2. In both cases it is enough to show that Q_1 is homeomorphic to a subset of R, and Q_2 is homeomorphic to a subset of R. It will be shown that Q_2 is homeomorphic to a subset of R. The proof of the fact that Q_1 is homeomorphic to a subset of R is similar. If Q_2 is empty then there is nothing to prove. Otherwise one can assume that $x_1 \notin Q_2$ and $x_2 \in Q_2$. Let $\{g_1, g_2, ..., g_n\}$ be a basis of N such that $g_1(x_1) = 1$, $g_i(x_1) = 0$ for each $i = 2, ..., n$, and let N' be the $(n-1)$-dimensional subspace of N generated by $\{g_2, ..., g_n\}$. Then since N has the property awc_1, it follows that no $g \neq 0$ in N' can have more than $(n-1)$ zeros in Q_2. So if Q_2 is not finite, then by Proposition 4.3, $\dim N|_{Q_2} = n-1 \geqslant 2$. Also since $\{x_1, x_2\}$ is N-totally dependent, it follows that $g(x_2) = 0$ for each $g \in N'|_{Q_2}$. Thus $N'|_{Q_2}$ is an $(n-1)$-dimensional subspace of $C(Q_2)$ that has the property awc_1, and $g(x_2) = 0$ for each $g \in N'|_{Q_2}$. Therefore, since $n-1 \geqslant 2$, it follows by Lemma 4.5 that it is enough to show that $N'|_{Q_2}$ is a weak Chebyshev subspace of $C(Q_2)$. By Theorem 1.5 it is enough to show that there is $\varepsilon = \pm 1$ such that for each $y_2 < \cdots < y_n$ in Q_2, it is always true that

$$\varepsilon \det[g_i(y_j)]_{i=2, j=2}^{n, n} \geqslant 0.$$

Since N is a weak Chebyshev subspace, then there is $\varepsilon = \pm 1$ such that for each $y_1 < y_2 < \cdots < y_n$ in Q

$$\varepsilon \det[g_i(y_j)]_{i=1, j=1}^{n, n} \geqslant 0.$$

Let $y_1 = x_1$, and choose any $y_2 < y_3 < \cdots < y_{n-1}$ in Q_2. Since $g_1(y_1) = 1$ and $g_i(y_1) = 0$ for $i = 2, ..., n$, it follows that

$$\varepsilon \det[g_i(y_j)]_{i=2, j=2}^{n,n} = \varepsilon \det[g_i(y_j)]_{i=1, j=1}^{n,n} \geqslant 0.$$

4.7. LEMMA. *Let Q be a compact totally ordered space. If $C(Q)$ contains a three-dimensional weak Chebyshev subspace that has the property* awc_1, *then Q is homeomorphic to a subset of R.*

Proof. If each subset of Q that consists of a three points is N-independent, then no $g \neq 0$ in N can have more than two zeros, so N is a three-dimensional Chebyshev subspace of $C(Q)$. Therefore, by Theorem 4.1, the set Q is homeomorphic to a subset of R.

Assume that there is at least one N-dependent subset of Q that consists of three points. Then there is at least one N-totally dependent nonempty subset of Q that contains at most three points.

If there is $x_0 \in Q$ such that $\{x_0\}$ is N-totally dependent, then $g(x_0) = 0$ for each $g \in N$. Thus by Lemma 4.5, the set Q is homeomorphic to a subset of R, and if there are $x_1 < x_2$ in Q such that $\{x_1, x_2\}$ is N-totally dependent, then by Lemma 4.6, the set Q is homeomorphic to a subset of R. Thus one may assume without loss of generality that there is at least one N-totally dependent subset of Q that contains exactly three points, and each N-totally dependent subset of Q that contains at most three points must contain exactly three points. Using Theorem 2.3, one can isolate the following two cases only:

Case 1. There are $x_1 < x_2 < x_3$ in Q such that $\{x_1, x_2, x_3\}$ is N-totally dependent, $Q = \{x \in Q; x \leqslant x_1\} \cup \{x_2\} \cup \{x \in Q; x \geqslant x_3\}$ and $\{x \in Q; x < x_1\} \neq \emptyset$, $\{x \in Q; x > x_3\} \neq \emptyset$.

Let $Q_1 = \{x \in Q; x \leqslant x_1\}$ and $Q_2 = \{x \in Q; x \geqslant x_3\}$. Then it is enough to show that both Q_1 and Q_2 are homeomorphic to subsets of R. To prove that Q_1 is homeomorphic to a subset of R, let $N' = \{g \in N; g(x_3) = 0\}$. Then N' is a two-dimensional subspace of $C(Q)$ and no $g \neq 0$ in N' can have more than two zeros in Q_1. Therefore, by Proposition 4.3 if Q_1 is not finite then $\dim N'|_{Q_1} = 2$. It will be shown that no $g \neq 0$ in N' can have more than one zero in Q_1. Indeed, if there is $g \neq 0$ in N' and $y_1 < y_2$ in Q_1 such that $g(y_1) = g(y_2) = 0$, then $\{y_1, y_2, x_3\}$ are the zeros of g. Thus the set $\{y_1, y_2, x_3\}$ is N-dependent, and since each N-totally dependent subset of Q that contains at most three points must contain three points, it follows that $\{y_1, y_2, x_3\}$ is N-totally dependent. But this contradicts Theorem 2.3 because $x_2 \in \{x \in Q; y_2 < x < x_3\}$ and $\{x \in Q; x > x_3\} \neq \emptyset$. Therefore $N'|_{Q_1}$ is a two-dimensional Chebyshev subspace of $C(Q_1)$. Thus by Theorem 4.1, the set Q_1 is homeomorphic to a subset of R. In the same way, one can show that Q_2 is homeomorphic to a subset of R. Therefore, Q is homeomorphic to a subset of R.

Case 2. There are two subsets A_1, A_2 each of which consists of at most two points of Q, $y_1 < y_2$ in Q such that $y_1 > x$ for each $x \in A_1$, $y_2 < x$ for each $x \in A_2$ and

$$Q = A_1 \cup \{x \in Q;\, y_1 \leqslant x \leqslant y_2\} \cup A_2.$$

Furthermore no subset of $\{x \in Q;\, y_1 \leqslant x \leqslant y_2\}$ that consists of at most three points is N-totally dependent.

In this case either the set $Q_0 = \{x \in Q;\, y_1 \leqslant x \leqslant y_2\}$ is finite or by Proposition 4.3 dim $N|_{Q_0} = 3$. If dim $N|_{Q_0} = 3$, then since each subset of Q_0 that consists of three points is N-independent, it follows that $N|_{Q_0}$ is a three-dimensional Chebyshev subspace of $C(Q_0)$. By Theorem 4.1, the set Q_0, and therefore the set Q, is homeomorphic to a subset of R.

4.8. Theorem. *Let Q be a compact totally ordered space. If $C(Q)$ contains a finite-dimensional weak Chebyshev subspace N of dimension at least three, and N has the property* awc$_1$, *then Q is homeomorphic to a subset of R.*

Proof. By induction. Let dim $N = n$. If $n = 3$ then by Lemma 4.7, the hypothesis is true. Assume that the hypothesis is true for $n-1 \geqslant 3$. It will be shown that it is true for n.

Let N be a n-dimensional weak Chebyshev subspace of $C(Q)$ that has the property awc$_1$. If each n points of Q are N-independent, then N is a Chebyshev subspace of $C(Q)$. Therefore by Theorem 4.1, the set Q is homeomorphic to a subset of R. If N is not Chebyshev subspace of $C(Q)$, then there are $x_1 < x_2 < \cdots < x_k$ in Q with $1 \leqslant k \leqslant n$ such that $\{x_1, ..., x_k\}$ is N-totally dependent. If $k = 1$, then by Lemma 4.5, the set Q is homeomorphic to a subset of R; if $k = 2$, then by Lemma 4.6, the set Q is homeomorphic to a subset of R. Thus one might assume that $k \geqslant 3$ and Q is not finite. By Theorem 2.3, there is $i_0 \in \{1, 2, ..., k\}$ such that x_{i_0} is an isolated point for Q. Let $Q_1 = \{x \in Q;\, x < x_{i_0}\}$ and $Q_2 = \{x \in Q;\, x > x_{i_0}\}$. Then Q_1 and Q_2 are compact and $Q = Q_1 \cup \{x_{i_0}\} \cup Q_2$. Therefore to prove that Q is homeomorphic to a subset of R, it is enough to show that Q_1 and Q_2 are both homeomorphic to subsets of R. If Q_2 is empty or finite, then it is homeomorphic to a subset of R. Otherwise one might assume that dim $N|_{Q_2} = n$. Let $N' = \{g \in N;\, g(x_{i_0}) = 0\}$. Then dim $N' = n-1$, and since N has the property awc$_1$, it follows that no $g \neq 0$ in N' can have more than $(n-1)$ zeros in Q_2. Thus $N|_{Q_2}$ is an $(n-1)$-dimensional subspace of $C(Q_2)$ that has the property awc$_1$. It will be shown that $N|_{Q_2}$ is a weak Chebyshev subspace of $C(Q_2)$. Let $\{g_2, ..., g_n\}$ be a basis for N' and let $g_1 \in N$ be such that $g_1(x_{i_0})=1$. Then $\{g_1, g_2, ..., g_n\}$ is a basis for N. Since N is a weak Chebyshev subspace of $C(Q)$, it follows that there is $\varepsilon = \pm 1$ such that for each $y_1 < y_2 < \cdots < y_n$ in Q;

$$\varepsilon \det[g_i(y_j)]_{i=1, j=1}^{n,n} \geqslant 0.$$

Let $y_2 < y_3 < \cdots < y_n$ be $(n-1)$ points of Q_2 and let $y_1 = x_{i_0}$. Then

$$\varepsilon \det[g_i(y_j)]_{i=2, j=2}^{n,n} = \varepsilon \det[g_i(y_j)]_{i=1, j=1}^{n,n} \geqslant 0.$$

Thus $N'|_{Q_2}$ is an $(n-1)$-dimensional weak Chebyshev subspace of $C(Q_2)$ that has the property awc$_1$. Therefore Q_2 is homeomorphic to a subset of R. In the same way one can show that Q_1 is also homeomorphic to a subset of R. So Q is homeomorphic to a subset of R.

4.9. THEOREM. *Let Q be a compact totally ordered space, and let N be an n-dimensional weak Chebyshev Z-subspace of $C(Q)$ such that $n \geqslant 3$. If the metric projection P_N has a continuous selection, then Q is homeomorphic to a subset of R.*

Proof. By Theorem 3.6, N has the property awc$_1$. Therefore by Theorem 4.8, the set Q is homeomorphic to a subset of R.

Another way of writing Theorem 4.9 is as follows:

4.10. THEOREM. *Let Q be a compact totally ordered space that is not homeomorphic to any subset of R, and let N be an n-dimensional weak Chebyshev Z-subspace of $C(Q)$. If $n \geqslant 3$ then the metric projection P_N has no continuous selection.*

Theorems 4.8 *and* 4.9 *need not be true if* dim $N = 2$. *The following example shows that there is a compact totally ordered space Q_0 that is not homeomorphic to any subset of R, and such that $C(Q_0)$ contains a two-dimensional weak Chebyshev Z-subspace which has the property* awc$_1$.

4.11. EXAMPLE. Let Q_0 be the set $([0, 1] \times \{0, 1\}) \backslash \{(0, 0), (1, 1)\}$, and let $\leqslant$ denote the lexicographic ordering on Q_0; that is, $(a, b) \leqslant (c, d)$ if and only if $a < c$ or $a = c$ and $b \leqslant d$. By Brown [2] the totally ordered space Q_0 is compact separable, and not homeomorphic to any subset of R. Furthermore, no $x \in Q_0$ is an isolated point.

Define g_1 and g_2 on Q_0 as follows:

$$g_1(x, y) = 1 \quad \text{for each} \quad (x, y) \in Q_0,$$

$$g_2(x, y) = x \quad \text{for each} \quad (x, y) \in Q_0.$$

Then $\{g_1, g_2\}$ is a subset of $C(Q_0)$. Let N be the two-dimensional subspace of $C(Q_0)$ generated by $\{g_1, g_2\}$. Then N is a Z-subspaced of $C(Q_0)$ and each $g \in N$ has at most one change of sign. By Theorem 1.5, N is a two-dimensional weak Chebyshev subspace of $C(Q_0)$. On the other hand, $g \neq 0$ in N has more than one zero in the set $\{(x, 0); x \in (0, 1)\}$ so no $g \neq 0$ in N has more than two zeros in Q. Thus N has the property awc_1.

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